proved differential amplifier, the accuracy of this method of measurement could be increased by an order of magnitude.

Other uses of this system could be Q measurements at millimeter-wave frequencies, thickness measurements of dielectric sheets, or dielectric constant measurements of known thickness sheets. This system has the further advantage that it may be used over the entire millimeter-wave spectrum by changing only the source and detectors to respond to the desired frequency.

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Some Characteristics of Alternating Gradient Optical Transmission Lines

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Abstract—**The effect of adding a negative lens between each pair of positive lenses of an optical transmission line is calculated. The negative lens reduces the ability of the transmission line to control the direction of the light beam. The changes in dominant mode spot size, allowed bending radius, critical bend periodicity, and sensitivity to random lateral lens displacements are computed for all ranges of lens spacings and focal lengths which are stable.**

Introduction

In SOME of the proposed light guidance methods which use gas or schlieren-type lenses it may be necessary to consider a system made of alternately positive and negative focal length lenses. For example, if tubular thermal gradient gas lenses [1]–[4] are used, which employ a continuous stream of gas flowing through them, it will be necessary to cool the gas periodically. The region where the gas is cooled will constitute a negative lens of possibly different power from the positive lens [5]. It is of interest, therefore, to consider how the periodic introduction of this negative lens affects the ability of the light guide to control the direction of the light beam.

In this paper, the effect of adding divergent lenses between each pair of positive lenses of a light guide will be considered. The changes in beam spot size, allowed bending radius of the guide, and stability of the guide to lateral lens displacements are calculated for any ratio of positive to negative lens power. Miller [6] previously calculated the stability conditions and some optimum design parameters for alternate gradient focusing when the power of the positive lenses and the power of the negative lenses are equal.

Beam Spot Size

Consider the transmission line shown in Fig. 1. A negative lens of focal length $-f/\alpha$ is placed between each pair of positive lenses of focal length f. The positive lenses are spaced $2\beta f$.

The properties of the dominant Gaussian mode of this transmission line can be analyzed by the ray matrix technique of Kogelnik [7]. The ray matrix is the transformation matrix for ray position and slope between the input and output planes. If the input plane is just to the right of a positive lens and the output plane is just to the right of the next positive lens, the ray matrix is

$$\begin{vmatrix} \beta\alpha + 1 & \beta f(2+\beta\alpha) \\ \dfrac{\alpha(1-\beta)-1}{f} & 1+\beta(\alpha-2-\beta\alpha) \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

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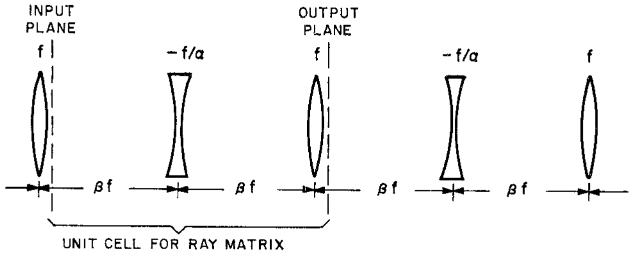

Fig. 1. Alternating gradient optical transmission line.

Rigrod [8] and Kogelnik [7] have pointed out that the lowest order Gaussian mode spot size w and the radius of curvature of the phase front R of the dominant mode just to the right of the positive lens can be found from the elements of the ray matrix. They show that

$$R = \frac{2B}{D - A}$$

$$\frac{\pi w^2}{\lambda} = \frac{2B}{\sqrt{4 - (A + D)^2}}$$

where A, B, C, and D are the matrix elements as shown, and λ is the wavelength.

If we substitute into these, we find

$$R = -2f$$

$$\frac{\pi w^2}{\lambda} = 2f\left[\frac{\beta(2 + \beta\alpha)}{(2 + \beta\alpha - 2\alpha)(2 - \beta)}\right]^{1/2}. \quad (1)$$

Figure 2 shows the normalized spot size, $[\pi/(2\lambda f)]^{1/2}w$, as a function of α and β.

The stability conditions for the transmission line as seen from (1) are

$$2\left(\frac{\alpha - 1}{\alpha}\right) < \beta < 2.$$

When $\alpha = 1$ this agrees with Miller [6]. When the negative lens becomes stronger than the positive lens ($\alpha > 1$), the line becomes unstable if the lenses are brought too close together. If $\alpha < 1$, the line is stable for $0 \leq \beta < 2$. However, when $\beta = 0$, Fig. 2 should not be used to be consistent with the theory of Gaussian modes [9]. If $\alpha = 1$, the line is not stable when $\beta = 0$ since the lenses then exactly cancel one another.

Figure 3 shows the shape of the dominant mode beam envelope as the strength of the negative lens is increased. When $\alpha = 0$, the spot size at the positive lens is such that it gives a beam waist midway between the two positive lenses. As the power of the negative lens increases, the spot size at the positive lens increases. If it were not for the negative lens, this beam would have a waist further from the positive lens (shown as dotted lines). The negative lens diverges the beam. As the negative lens power continues to increase, the spot size increases and the would-be beam waist moves further out. In the limit the spot size is infinite and the would-be beam waist occurs at $2f$ from the positive lens.

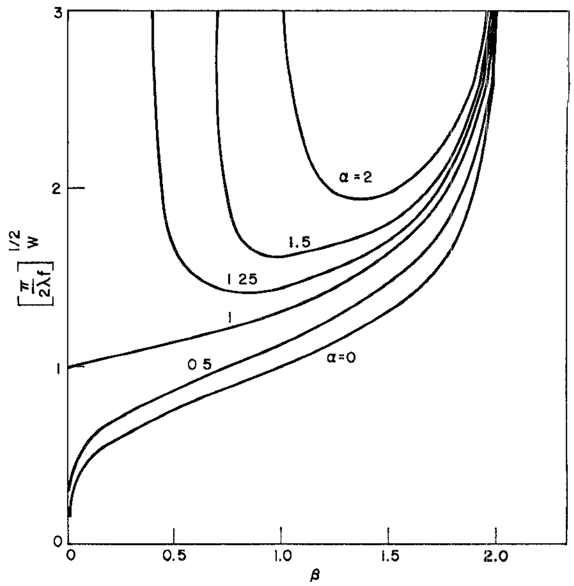

Fig. 2. Normalized dominant Gaussian mode spot size.

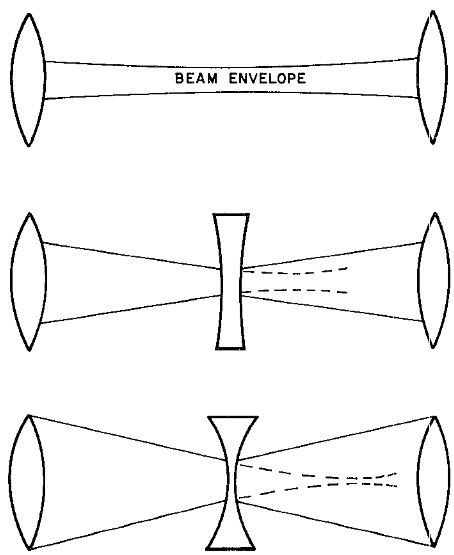

Fig. 3. Effect on beam envelope of increasing the negative lens power.

Bending Radius of the Light Guide

An important consideration for a light guide is the ability of the guide to control the beam direction. That is, how much can the axis of the light guide be bent without losing the light beam? This is closely related to the beam spot size as pointed out by Miller [10]. It is of interest to compute how the introduction of a negative lens between each pair of positive lenses affects the allowed bending radius of the light guide.

This can be computed using geometrical optics since it has been shown that in the paraxial approximation the center of a Gaussian beam follows the laws of geometric optics [11], [12]. For simplicity, a two-dimensional analysis will be made. This is applicable to the three-dimensional case since it has been shown [13] that the two transverse dimensions are independent and can each be treated separately.

Consider a sequence of positive and negative lenses placed on a curved axis of radius R, as shown in Fig. 4. Plane "a" is just to the right of the positive lens "a," plane "b" is just to the right of the negative lens "b," and plane "c" is just to the right of the next positive lens "c." Let r be the position of the ray measured from the lens center, and r' be the ray slope measured from the line joining the lenses.

Following Pierce [14],

$$r_b = r_a + \beta f r_a'$$

$$r_b' = r_a' + \Phi + \frac{\alpha r_b}{f}$$

$$r_c = r_b + \beta f r_b'$$

$$r_c' = r_b' + \Phi - \frac{r_c}{f}.$$

Combining these, and defining $r_a = r_n$, $r_c = r_{n+1}$, and letting $\Phi \approx \beta f/R$,

$$r_{n+1} = r_n(1 + \beta\alpha) + r_n'\beta f(2 + \beta\alpha) + \frac{(\beta f)^2}{R}. \qquad (2)$$

From this the difference equation can be derived as

$$r_{n+2} - [2 - 2\beta(1 - \alpha) - \beta^2\alpha]r_{n+1} + r_n = \frac{\beta^2 f^2}{R}[4 + \beta\alpha]. \quad (3)$$

Figure 5 shows a straight section of guide followed by a curved section. The first lens on the curved section is labeled "0." Let us assume a ray incident upon lens "0" at r_0 with angle γ measured with respect to the straight guide axis. The initial conditions are therefore

$$r_0 = r_0$$

$$r_0' = \gamma + \frac{\beta f}{2R} - \frac{r_0}{f}$$

$$r_1 = r_0(1 + \beta\alpha - 2\beta - \beta^2\alpha) + \gamma\beta f(2 + \beta\alpha) + \frac{(\beta f)^2(4 + \beta\alpha)}{2R}.$$

Using the method of Hirano and Fukatsu [13], the solution to the difference equation (3) for the ray on the curved guide can be written as

$$r_{n+1} = \frac{\beta^2 f^2}{R}(4 + \beta\alpha)\sum_{k=0}^{n}\frac{\sin k\theta}{\sin\theta} + r_1\frac{\sin(n+1)\theta}{\sin\theta} - r_0\frac{\sin n\theta}{\sin\theta}$$

where

$$\theta = \cos^{-1}\left(1 - \beta + \beta\alpha - \frac{\beta^2\alpha}{2}\right).$$

After substitution of the initial conditions and use of trigonometric identities, this can be written as

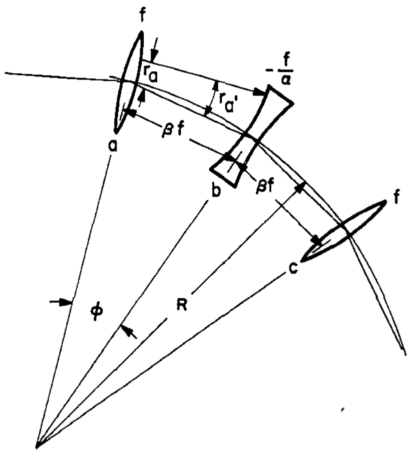

Fig. 4. Curved axis transmission line.

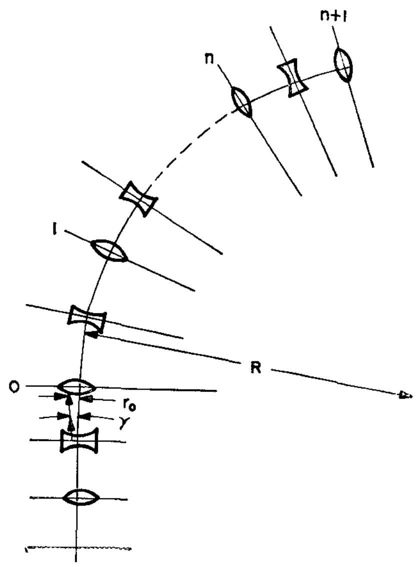

Fig. 5. Beam injection into curved transmission line.

$$r_n = \frac{\beta^2 f^2(4 + \beta\alpha)}{2R(1 - \cos\theta)} + \left[\frac{\beta^2 f^2(4 + \beta\alpha)}{2R(1 - \cos\theta)} + r_0\right]\cos n\theta + \left[\frac{\beta(2 + \beta\alpha)\left(\gamma f - \frac{r_0}{2}\right)}{\sin\theta}\right]\sin n\theta. \qquad (4)$$

Equation (4) shows how the beam oscillates about the axis of the guide when the axis is curved. If we assume the beam enters on axis ($r_0 = \gamma = 0$), the maximum amplitude of the beam oscillation is

$$r_{max} = \frac{\beta^2 f^2(4 + \beta\alpha)}{R(1 - \cos\theta)} = \frac{\beta f^2(4 + \beta\alpha)}{R\left(1 - \alpha + \frac{\beta\alpha}{2}\right)}.$$

If the r_{max} is defined by how much the beam is allowed to wander in the guide, then the allowed bending radius of the guide can be found from

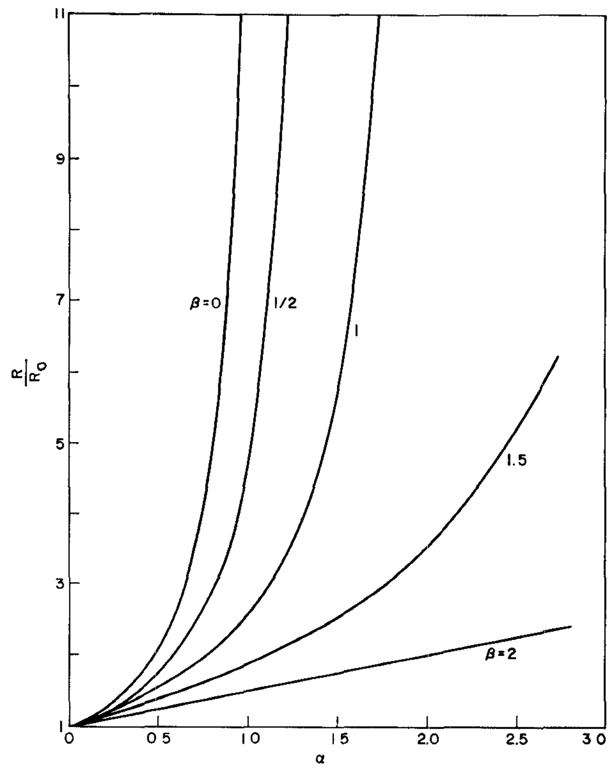

Fig. 6. Effect on allowed bending radius.

$$R = \frac{\beta f^2(4+\beta\alpha)}{r_{\max}\left(1-\alpha+\frac{\beta\alpha}{2}\right)}.$$

For the case of no divergent lenses ($\alpha=0$), the allowed bending radius is

$$R_0 = \frac{4\beta f^2}{r_{\max}}$$

which agrees with Marcuse [15]. The $r_{\max}$ will be determined by the lens diameters. If we assume a constant positive lens diameter, the effect on the bending radius of adding the divergent lenses can be found from

$$\frac{R}{R_0} = \frac{4+\beta\alpha}{4\left(1-\alpha+\frac{\beta\alpha}{2}\right)}.$$

This gives a measure of the extent to which the divergent lenses reduce the light guide's ability to control the direction of the light beam. Figure 6 shows R/R_0 as a function of α and β.

Stability Due to Lateral Lens Displacements

Any lateral displacement of a lens from the straight line axis will deflect the light beam and cause it to follow an oscillatory path thereafter. If the amplitude of the oscillatory path is large, the beam will hit the edge of a lens and be lost.

For correlated lateral lens displacements, the amplitude of the displacements which have periodicities near the critical periodicity must be kept very low to prevent loss of the beam [16]–[18]. The critical periodicity is determined by the lens focal length and spacing.

For random lateral lens displacements, the expected beam deviation at the output is proportional to the square root of the number of lenses and the rms lens displacement [13], [19]. This beam deviation from the line axis can become large when the number of lenses is large.

The construction and laying tolerances of a transmission line will be largely determined by the effects of lateral lens displacements. If it is necessary to introduce a negative lens periodically, it is of interest to know how these tolerances are affected. The negative lens will change the critical periodicity and will change the sensitivity to random lens displacements.

The homogeneous portion of the difference equation given as (3) has solutions of the form

$$r_n = A \begin{matrix}\sin\\ \cos\end{matrix}\left\{n\cos^{-1}\left(1-\beta+\beta\alpha-\frac{\beta^2\alpha}{2}\right)\right\}.$$

If the nominal axis of the guide is straight, the lens displacements act as a driving term in the difference equation. If the driving terms have the same periodicity as the solutions to the homogeneous equation, one would expect large beam deviations. The critical period Λ_c can therefore be defined as

$$\frac{\Lambda_c}{L} = \frac{2\pi}{\cos^{-1}\left(1-\beta+\beta\alpha-\frac{\beta^2\alpha}{2}\right)}$$

where L = spacing between the positive lenses. Figure 7 shows Λ_c/L as a function of α and β. For a fixed lens spacing the critical period becomes larger as the strength of the negative lens increases. If the transmission line has a serpentine bend or random bends whose periods are near Λ_c, the beam deviation will become large and the beam will be lost when it hits the lens edges.

The sensitivity to random lens displacements can be found by using the same method as Hirano and Fukatsu [13]. Using the notation shown in Fig. 8 we can write

$$r_b = r_a + \beta f r_a'$$

$$r_b' = r_a' + \frac{\alpha}{f}(r - d_b)$$

$$r_c = r_b + \beta f r_b'$$

$$r_c' = r_b' - \frac{1}{f}(r_c - t_c)$$

where d_b = distance from the center of the divergent lens to the transmission line axis, and t_c = distance from the center of the convergent lens to the transmission line axis. Both d_b and t_c are positive if the lens center is above the transmission line axis.

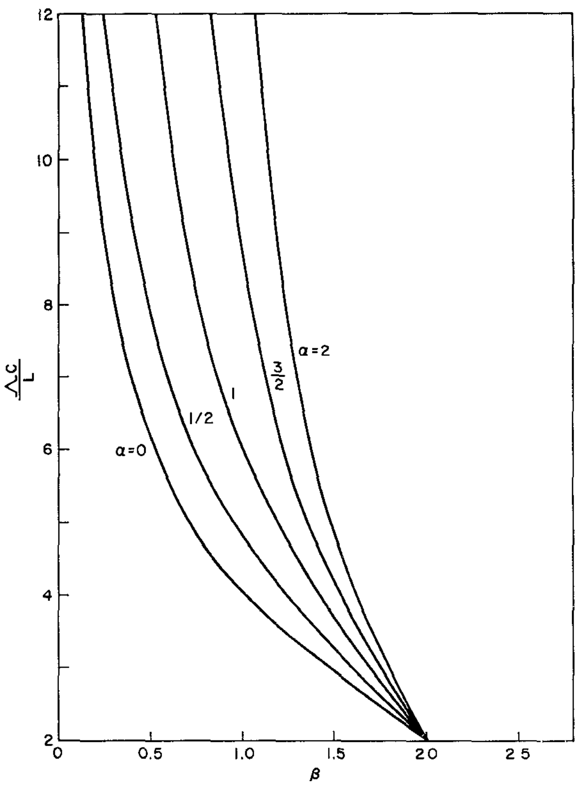

Fig. 7. Effect on critical period.

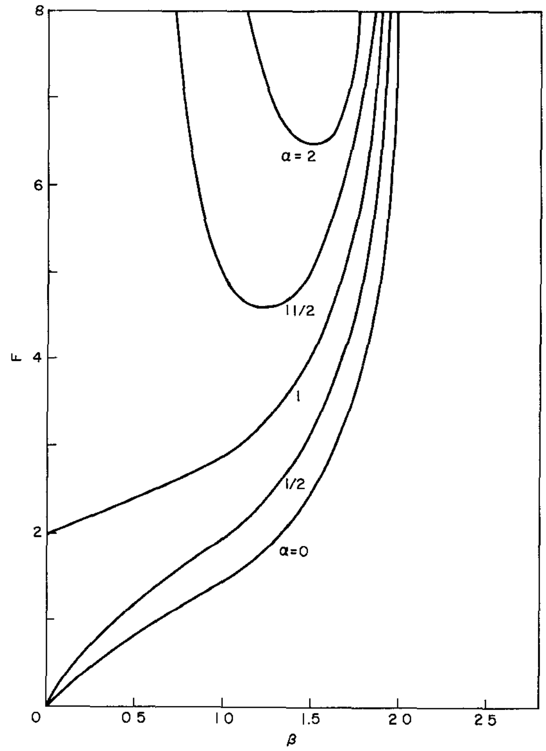

Fig. 9. Increase in sensitivity to random lateral lens displacement.

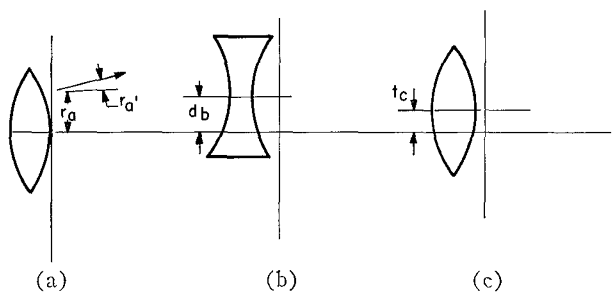

Fig. 8. Lateral lens displacement.

If t_n and d_n are the displacements of the lenses to the left of the nth plane, then the difference equation can be written as

$$r_{n+2} - [2 - 2\beta(1-\alpha) - \beta^2\alpha]r_{n+1} + r_n = t_{n+1}(2\beta + \beta^2\alpha) - \beta\alpha(d_{n+1} + d_{n+2}).$$

Using the same technique as in Section III, the solution, if $r_0 = r_1 = 0$, is

$$r_n = \frac{\beta}{\sin\theta}\sum_{k=0}^{n-1}[t_{k+1}(2+\beta\alpha) - \alpha(d_{k+1} + d_{k+2})] \cdot \sin(n-1-k)\theta.$$

For random displacements we assume

$$\langle t_k\rangle = \langle d_k\rangle = 0$$
$$\langle t_k t_m\rangle = \langle d_k d_m\rangle = 0 \quad k \neq m$$
$$\langle t_k d_m\rangle = 0 \quad \text{for all } k$$
$$\langle t_k^2\rangle = \langle d_k^2\rangle = \sigma^2$$

where the symbols $\langle\ \rangle$ denote the expected values of random variables.

From this one can compute that

$$\sqrt{\langle r_{n+1}^2\rangle} = \frac{\beta\sigma}{\sin\theta}\left\{[2\alpha^2 + (2+\beta\alpha)^2] \cdot \left[\frac{n}{2} + \frac{1}{4} - \frac{\sin(2n-1)\theta}{4\sin\theta} - \frac{\cos 2n\theta}{2}\right] + \alpha^2\left[n\cos\theta - \frac{\cos n\theta \sin n\theta}{\sin\theta}\right]\right\}^{1/2}.$$

In the case where n is large and we are sufficiently far from an unstable arrangement ($\sin\theta \neq 0$), then

$$\sqrt{\langle r_{n+1}^2\rangle} \approx F\sigma\sqrt{n} \tag{5}$$

where

$$F = \left\{\beta\,\frac{2(\alpha^2+1) + \beta\alpha(\alpha^2 - \alpha + 2) + \dfrac{\beta^2\alpha^2}{2}(1-\alpha)}{(2-\beta)\left[1 + \alpha(\beta - 1) + \dfrac{\alpha^2\beta}{4}(\beta - 2)\right]}\right\}^{1/2}.$$

When $\alpha = 0$, this is the same as the F_1 defined by Hirano and Fukatsu [13].

Hence, the expected value of the output beam displacement still increases as the square root of n. The introduction of the negative lens increases the sensitivity to random lens displacements by increasing F.

The function F is plotted in Fig. 9. It should be pointed out that the number n actually represents $2n$

lenses, i.e., n positive lenses and n negative lenses. There are therefore $2n$ lenses, all of which have random lateral displacements with an rms value of σ.

For the case of positive lenses spaced confocally and equal power negative lenses ($\alpha=\beta=1$), then

$$\sqrt{\langle r_{n+1}^2 \rangle} = 2.82\sigma\sqrt{n}.$$

In the case of no negative lenses ($\alpha=0$, $\beta=1$), then

$$\sqrt{\langle r_{n+1}^2 \rangle} = 1.41\sigma\sqrt{n}.$$

In the first case the expected deviation of the output beam is twice that of the second case, but there are twice as many lenses to align in the first case. If this increase in lateral sensitivity were due only to the increased number of lenses, one would expect an increase of only $\sqrt{2}$. The additional factor of $\sqrt{2}$ is due to the reduced focusing properties of the line.

Summary

As expected, the addition of the negative lenses reduces the ability of the transmission line to control the light beam. However, if the power of the negative lenses is kept equal to or less than the power of the positive lenses, the reduction in guiding ability is not too severe.

For example, consider a transmission line of positive lenses spaced confocally ($\beta=1$) and add negative lenses of the same power ($\alpha=1$):

1) The spot size at the positive lens is increased by 1.315.
2) The allowed bending radius is increased by 2.5.
3) The critical bending period is increased by 1.5.
4) The sensitivity to random lateral lens displacements is increased by 2.

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Mode Conversion in Tapered Waveguides At and Near Cutoff

C. C. H. TANG

Abstract—The coupling coefficient between the TE_{11} mode and the TM_{11} mode in tapered circular waveguides is derived, and at cutoff frequency it tends to approach an infinity of the order of $0^{-1/4}$. It is surprising to discover that the corresponding coupling coefficient between the TE_{10} mode and the TM_{12} mode in tapered rectangular waveguides approaches instead a zero of the order of $0^{1/4}$ at cutoff frequency. Accordingly, for the modes concerned, the choice of using circular or square waveguides as tapers for transition at and near cutoff frequency is significant in reducing mode conversion level. At and near cutoff frequency a "synthesized" square taper is better in that it is shorter than a "synthesized" circular taper for the same mode conversion levels. On the other hand, for frequencies far away from cutoff the choice is insignificant.

Design procedures for "synthesized" waveguide tapers at and near cutoff are presented, and the results of measurements are in agreement with the theoretical calculations.

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